A representation of the Zeta function.

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Let T be a self-adjoint operator in some Hilbert space H. Assume that $(Tu, u) \geq ||u||^2$ for $u \in \mathscr{D}(T)$ and $T^{-1} \in C_q(H)$ for some $q > 0$. Note that every eigenvalue of T is greater then 1.

Define the zeta function ζ_T of T by

$$\zeta_T(s) := \sum_{\lambda \in spec\, T \setminus \{0\}} \lambda^{-s}, \qquad Re\, s > q.$$

We will now deduce the following representation of $\zeta_T(s)$ which is true for $Res > n$:

$$\binom{n-1-s}{n-1} \zeta_T(s) = \frac{\sin \pi s}{\pi} \int\limits_1^\infty z^{n-1-s} tr(z-T)^{-n} dz + \tilde{\zeta}_T(s),$$

where $\tilde{\zeta}_T(s)$ is entire function.

We introduce the following paths on a complex plane, defined for $0 < r \leq R \leq \infty$ and $0 \leq a < \pi$:

$$L_{r,R,a} := \{te^{ia} : r \leq t \leq R\},$$

$$C_{r,a} := \{re^{i\alpha} : |\alpha| \leq a\}.$$

Let γ be the closed path on a complex plane consisting of four pieces: $C_{1/2,a}$ passed clockwise, $C_{R,a}$ counterclockwise and two line segments $L_{1/2,R,a}$ and $L_{1/2,R,-a}$.

Now using Cauchy formula for a function λ^{-s} and the path γ we get

$$\lambda^{-s} = \frac{1}{2\pi i} \int\limits_\gamma \frac{z^{-s}}{z-\lambda} dz = \frac{1}{2\pi i}(- \int\limits_{L_{1/2,R,a}} - \int\limits_{C_{1/2,a}} + \int\limits_{L_{1/2,R,-a}} + \int\limits_{C_{R,a}}) \frac{z^{-s}}{z-\lambda} dz.$$

Denote the integrals I_1, I_2, I_3, I_4 correspondingly. Consider the sum $I_1 + I_3$

and let a tend to π then use the definition of a line integral

$$\lim_{a\to\pi}(I_1+I_3)$$

$$\begin{aligned} &= -\int\limits_{1/2}^{R} \frac{(te^{i\pi})^{-s}}{te^{i\pi}-\lambda} e^{i\pi} dt + \int\limits_{1/2}^{R} \frac{(te^{-i\pi})^{-s}}{te^{-i\pi}-\lambda} e^{-i\pi} dt \\ &= (e^{i\pi s} - e^{-i\pi s}) \int\limits_{1/2}^{R} \frac{t^{-s}}{t-\lambda} dt \\ &= 2i \sin \pi s \int\limits_{1/2}^{R} \frac{t^{-s}}{t-\lambda} dt \end{aligned} \tag{1}$$

Now consider I_4. Since $Re^{i\alpha}-\lambda$ for $-\pi \leq \alpha \leq \pi$ lies on a circle centered at $-\lambda$ with radius $R > \lambda > 1$ we get the inequality $|R-\lambda| \leq |Re^{i\alpha}-\lambda|$. So we can estimate the integral

$$|I_4| \leq |i \int\limits_{-\pi}^{\pi} \frac{(Re^{i\alpha})^{1-s}}{Re^{i\alpha}-\lambda} d\alpha| \leq |\frac{2\pi R^{1-s}}{R-\lambda}| = |\frac{2\pi R^{-s}}{1-\lambda/R}|$$

Since $Re\, s > 0$

$$\lim_{R\to\infty} |I_4| = 0.$$

It remains to calculate the integral I_2.

$$I_2 = -2\pi i Res_{z=0}(\frac{z^{-s}}{z-\lambda}) = -2\pi i \frac{1}{(Res-1)!} \lim_{z\to 0} \frac{d^{Res}}{dz^{Res}} \Big((z)^{Res} \frac{z^{-s}}{z-\lambda}\Big) = \text{entire function of s.}$$

Summarize the integrals:

$$\lambda^{-s} = \lim_{R\to\infty} \lim_{a\to\pi} (I_1+I_2+I_3+I_4) = \frac{\sin \pi s}{\pi} \int\limits_{1/2}^{\infty} \frac{t^{-s}}{t-\lambda} dt + \text{entire function.}$$

Since $\lambda > 1$ there is no singularities between $1/2$ and 1

$$\lambda^{-s} = \frac{\sin \pi s}{\pi} \int\limits_{1}^{\infty} \frac{t^{-s}}{t-\lambda} dt + \text{entire function}$$

Now use integration by parts $n-1$ times for $n < Re\,s$

$$\lambda^{-s} = \frac{\sin \pi s}{s} \int_1^R \frac{t^{-s}}{t-\lambda} dt$$

$$= \frac{sin\pi s}{s} \frac{t^{-s+1}}{-s+1} (t-\lambda)^{-1}|_1^\infty + \frac{\sin \pi s}{s} \int_1^\infty \frac{t^{-s+1}}{-s+1} (t-\lambda)^{-2} dt$$

$$= \frac{sin\pi s}{s(-s+1)} (1-\lambda)^{-1} + \frac{\sin \pi s}{s} \int_1^\infty \frac{t^{-s+1}}{-s+1} (t-\lambda)^{-2} dt$$

$$= \frac{\sin \pi s}{s} \int_1^\infty \frac{t^{-s+2}}{(-s+1)(-s+2)} 2(t-\lambda)^{-3} dt + \text{entire function}$$

...

$$= \frac{\sin \pi s}{s} \int_1^\infty \frac{t^{-s+n-1}}{(-s+1)(-s+2)...(-s+n-1)} (n-1)!(t-\lambda)^{-n} dt + \text{ent. f.}$$

Finally we get

$$\zeta_T(s) = \frac{\sin \pi s}{s} \frac{(n-1)!}{(-s+1)(-s+2)...(-s+n-1)} \int_1^\infty t^{-s+n-1} tr(t-T)^{-n} dt + \text{ent. f.}$$

To obtain the desired representation we denote the entire function by $\tilde{\zeta}_T(s)$

$$\binom{n-1-s}{n-1} \zeta_T(s) = \frac{\sin \pi s}{\pi} \int_1^\infty z^{n-1-s} tr(z-T)^{-n} dz + \tilde{\zeta}_T(s),$$